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Some properties of the c-nilpotent multiplier of Lie algebras

Ali Reza Salemkar *, Zahra Riyahi

Faculty of Mathematical Sciences, Shahid Beheshti University, G.C., Tehran, Iran

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ABSTRACT

In this article, we give a sufficient condition for the c-nilpotent multiplier of a Lie algebra to be finite dimensional. Also, we show that the c-nilpotent multipliers of perfect Lie algebras are isomorphic.

1. Introduction and preliminary

Throughout this article, all Lie algebras are considered over some fixed field Λ and $[\ ,\ ]$ denotes the Lie bracket. Let L be a Lie algebra presented as the quotient of a free Lie algebra F by an ideal R. Then the *c-nilpotent multiplier* of L, $c \geqslant 1$, is defined to be the abelian Lie algebra $\mathcal{M}^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F)$, where $\gamma_{c+1}(F)$ denotes the $(c+1)$-th term of the lower central series of F and $\gamma_1(R, F) = R$, $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$ (see [7]). The Lie algebra $\mathcal{M}^{(1)}(L) = \mathcal{M}(L)$ is more known as the *Schur multiplier* of L (see [2,3,6] or [8] for more information on the Schur multiplier of Lie algebras). One may check that $\mathcal{M}^{(c)}(L)$ is independent of the choice of the free presentation of L. Furthermore, if we set $\gamma^*_{c+1}(L) = \gamma_{c+1}(F)/\gamma_{c+1}(R, F)$, then it is readily deduced from the short exact sequence

$$0 \longrightarrow \mathcal{M}^{(c)}(L) \longrightarrow \gamma^*_{c+1}(L) \longrightarrow \gamma_{c+1}(L) \longrightarrow 0$$

* Corresponding author.
E-mail address: salemkar@sbu.ac.ir (A.R. Salemkar).

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and the invariance of $\mathcal{M}^{(c)}(L)$ that $\gamma_{c+1}^*(L)$ is an invariant of L. It is obvious that the image of the canonical homomorphism $\gamma_{c+1}^*(L) \longrightarrow \gamma_c^*(L)$ is ideal in $\gamma_c^*(L)$, and $\gamma_{c+1}^*(L) = 1$ if and only if L is nilpotent of class c and $\mathcal{M}^{(c)}(L) = 0$.

A Lie algebra L is said to be *c-capable* if there exists a Lie algebra K with $L \cong K/Z_c(K)$, where $Z_c(K)$ is the c-th centre of K. Evidently, L is 1-capable if and only if it is an inner derivation Lie algebra, and L is c-capable ($c \geqslant 2$) if and only if it is an inner derivation Lie algebra of a $(c-1)$-capable Lie algebra. Now, we define $Z_c^*(L)$ to be the smallest ideal T of L such that L/T is c-capable. It is obvious that $Z_c^*(L)$ is a characteristic ideal of L contained in $Z_c(L)$, and $Z_c^*(L/Z_c^*(L)) = 0$.

It has been shown in [7] that the dimension of c-nilpotent multiplier of a finite dimensional Lie algebra is finite. In this article, we extend this result by proving the following

Theorem A. *Let L be a Lie algebra.*

(i) *If $L/Z_c^*(L)$ is finite dimensional, then both Lie algebras $\gamma_{c+1}^*(L)$ and $\mathcal{M}^{(c)}(L)$ are finite dimensional.*
(ii) *If $L/Z_c^*(L)$ is nilpotent (resp. solvable), then $\gamma_{c+1}^*(L)$ is nilpotent (resp. solvable).*

In general, the c-nilpotent multipliers of an arbitrary Lie algebra are not necessary isomorphic. For example, if L is a finite dimensional abelian Lie algebra then [7, Proposition 1.2] shows that $\mathcal{M}^{(c)}(L) \ncong \mathcal{M}^{(d)}(L)$ whenever $c \neq d$. In the next result, we prove that c-nilpotent multipliers of perfect Lie algebras are indeed isomorphic to the Schur multipliers.

Theorem B. *Let L be a prefect Lie algebra. Then the canonical homomorphisms*

$$\gamma_{c+1}^*(L) \xrightarrow{\cong} \gamma_2^*(L) \quad \text{and} \quad \mathcal{M}^{(c)}(L) \xrightarrow{\cong} \mathcal{M}(L)$$

are isomorphisms for $c \geqslant 1$.

To prove the above results we need to recall and develop some details on crossed modules and exterior products. A *crossed module* is a homomorphism of Lie algebras $\lambda : M \longrightarrow L$ with a Lie algebra action $(l, m) \longmapsto {}^l m$ of L on M satisfying (i) $\lambda({}^l m) = [l, \lambda(m)]$, (ii) ${}^{\lambda(m)}m' = [m, m']$, for all $m, m' \in M$, $l \in L$. If M is an ideal of L, then the inclusion map $M \hookrightarrow L$ is a crossed module. Given a free presentation $L \cong F/R$, one readily verifies

Lemma 1.1. *The canonical homomorphism $\mu_c : \gamma_c^*(L) \longrightarrow L$ is a crossed module in which an element $l \in L$ acts on an element $\bar{f} = f + \gamma_c(R, F)$ in $\gamma_c^*(L)$ by ${}^l\bar{f} = [\dot{l}, \bar{f}]$, where $\dot{l}$ is any lift of l in $F/\gamma_c(R, F)$.*

Let $\lambda : M \longrightarrow K$ and $\mu : L \longrightarrow K$ be two crossed modules. There are actions of M on L and of L on M given by ${}^m l = {}^{\lambda(m)}l$ and ${}^l m = {}^{\mu(l)}m$. We take M (and L) to act on itself by Lie multiplication. The *non-abelian exterior product* $M \wedge L$ is defined in [4] as the Lie algebra generated by the symbols $m \wedge l$ ($m \in M$, $l \in L$) subject to the relations

$$\begin{aligned}
&c(m \wedge l) = cm \wedge l = m \wedge cl, && {}^m m' \wedge l = m \wedge {}^{m'} l - m' \wedge {}^m l,\\
&(m + m') \wedge l = m \wedge l + m' \wedge l, && m \wedge {}^l l' = {}^{l'} m \wedge l - {}^l m \wedge l',\\
&m \wedge (l + l') = m \wedge l + m \wedge l', && [(m \wedge l), (m' \wedge l')] = -{}^l m \wedge {}^{m'} l',\\
&m \wedge l = 0 \quad \text{whenever } \lambda(m) = \mu(l)
\end{aligned}$$

for all $c \in \Lambda$, $m, m' \in M$ and $l, l' \in L$.

Any Lie algebra L acts on itself by Lie multiplication and so we can always form the exterior product $L \wedge L$. In [5], it is shown that the *commutator map* $\kappa_L : L \wedge L \longrightarrow L$ defined on generators by

$l_1 \wedge l_2 \longmapsto [l_1, l_2]$, together with the action of L on $L \wedge L$ given by ${}^l(l_1 \wedge l_2) = [l, l_1] \wedge l_2 + l_1 \wedge [l, l_2]$, is a crossed module. One thus gets the triple exterior product $(L \wedge L) \wedge L$, and applying this process gives $\bigwedge^{c+1} L = (\cdots((L \wedge L) \wedge L) \wedge \cdots \wedge L)$, $c \geqslant 1$, involving $(c+1)$ copies of L. Note that the image of κ_L is equal to the derived subalgebra of L, L^2, and its kernel is a central subalgebra of $L \wedge L$, which is isomorphic to $\mathcal{M}(L)$.

The following results are useful in our investigation.

Lemma 1.2. *Let L be a Lie algebra and $c \geqslant 1$. Then*:

(i) *There is an epimorphism*

$$\kappa : \gamma_c^*(L) \wedge L \longrightarrow \gamma_{c+1}^*(L), \quad x \wedge y \longmapsto [\widetilde{x}, \widetilde{y}\,]$$

where $\widetilde{x}$ and $\widetilde{y}$ are lifts in $F/\gamma_{c+1}(R, F)$ of $x \in \gamma_c^*(L)$ and $y \in L$.

(ii) *The Lie algebra $\gamma_{c+1}^*(L)$ is a homomorphic image of $\bigwedge^{c+1} L$.*

Proof. The part (i) is clear and the part (ii) is a straightforward consequence of [7, Proposition 1.4(i)]. $\square$

Proposition 1.3. *(See [4].) For any Lie algebra L, the exterior product $L \wedge L$ is isomorphic to $\gamma_2^*(L)$.*

2. Proof of theorems

To prove Theorem A, we first present some different forms for the ideal $Z_c^*(L)$ of a Lie algebra L.

Proposition 2.1. *The ideal $Z_c^*(L)$ of a Lie algebra L is the intersection of all subalgebras of the form $\theta(Z_c(L))$, where $\theta : K \longrightarrow L$ is an epimorphism with $\ker\theta \subseteq Z_c(K)$.*

Proof. Set $A = \bigcap\{\theta(Z_c(L)) \mid \theta : K \longrightarrow L$ is an epimorphism with $\ker\theta \subseteq Z_c(K)\}$. By the definition of $Z_c^*(L)$, there exists a Lie algebra K together with an epimorphism $\theta : K \longrightarrow L/Z_c^*(L)$ such that $\ker\theta = Z_c(K)$. Suppose $H = \{(l, k) \in L \oplus K \mid \theta(k) = l + Z_c^*(L)\}$ and $\phi : H \longrightarrow L$ denotes the projective map. It is readily verified that ϕ is an epimorphism with $\ker\phi \subseteq Z_c(H)$ and $\phi(Z_c(H)) \subseteq Z_c^*(L)$. It therefore follows that $A \subseteq Z_c^*(L)$. To prove the reverse containment, we first show that if $\{N_i \mid i \in I\}$ is a family of ideals of the Lie algebra L such that each L/N_i is c-capable, then so is $L/\bigcap_{i\in I} N_i$. For each $i \in I$, let $0 \longrightarrow Z_c(K_i) \longrightarrow K_i \xrightarrow{\theta_i} L/N_i \longrightarrow 0$ indicate the assumption that L/N_i is c-capable. Put $N = \bigcap_{i\in I} N_i$ and $K = \{(k_i) \in \prod_{i\in I} K_i \mid \exists l \in L$ such that $\theta_i(k_i) = l + N_i\ \forall i \in I\}$, where $\prod_{i\in I} K_i$ denotes the Cartesian product of the Lie algebras K_i. One may see that $Z_c(K) = \prod_{i\in I} Z_c(K_i)$. For each $l \in L$, we can choose elements $k_{l,i} \in K_i$ such that $\theta_i(k_{l,i}) = l + N$. Consequently, $k_l = (k_{l,i}) \in K$ and the map $L/N \longrightarrow K/Z_c(K)$ given by $l + n \longmapsto k_l + Z_c(K)$ is an isomorphism. The conclusion is that L/N is c-capable.

Now, let $\eta : B \longrightarrow L$ be an epimorphism with $\ker\eta \subseteq Z_c(B)$. Using the isomorphism $L/\eta(Z_c(B)) \cong B/Z_c(B)$ and the assertion above, we conclude that L/A is c-capable and thus $Z_c^*(L) = A$, as required. $\square$

Using the above proposition, we obtain another representation of $Z_c^*(L)$ by free presentations as follows:

Corollary 2.2. *Let $0 \longrightarrow R \longrightarrow F \xrightarrow{\pi} L \longrightarrow 0$ be a free presentation of a Lie algebra L. Then $Z_c^*(L) = \overline{\pi}(Z_c(F/\gamma_{c+1}(R, F)))$, where $\overline{\pi}$ is the natural epimorphism induced by π.*

Proof. Let $\theta : K \longrightarrow L$ be an epimorphism with $\ker\theta \subseteq Z_c(K)$. As F is free, there exists a homomorphism $\beta' : F \longrightarrow K$ such that $\theta\beta' = \pi$. It is easily checked that $\beta'(R) \subseteq \ker\theta$ and $\beta'(\gamma_{c+1}(R, F)) = 0$. Hence β' induces a homomorphism $\beta : F/\gamma_{c+1}(R, F) \longrightarrow K$ such that the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & \frac{R}{\gamma_{c+1}(R,F)} \arrow[r] \arrow[d, "\beta_1"'] & \frac{F}{\gamma_{c+1}(R,F)} \arrow[r, "\bar{\pi}"] \arrow[d, "\beta"'] & L \arrow[r] \arrow[d, "1_L"'] & 0 \\ 0 \arrow[r] & \ker\theta \arrow[r] & K \arrow[r, "\theta"] & L \arrow[r] & 0, \end{tikzcd}$$

where β_1 is the restriction of β to $R/\gamma_{c+1}(R, F)$. Obviously, $K = \ker\theta + \operatorname{Im}\beta$ and hence $\beta(Z_c(F/\gamma_{c+1}(R, F))) \subseteq Z_c(K)$. One deduces that $\bar{\pi}(Z_c(F/\gamma_{c+1}(R, F))) \subseteq \theta(Z_c(K))$. Now, the result follows from Proposition 2.1 $\square$

In the following, we show that $Z_c^*(L)$ is the largest ideal of L such that the Lie algebras $\gamma_{c+1}^*(L)$ and $\gamma_{c+1}^*(L/Z_c^*(L))$ are isomorphic.

Proposition 2.3. *Let L be a Lie algebra with an ideal N. Then $N \subseteq Z_c^*(L)$ if and only if the quotient homomorphism $L \longrightarrow L/N$ induces an isomorphism $\gamma_{c+1}^*(L) \xrightarrow{\cong} \gamma_{c+1}^*(L/N)$.*

Proof. Let $\gamma_{c+1}^*(L)$ and $\gamma_{c+1}^*(L/N)$ be defined in terms of free presentations $L \cong F/R$ and $L/N \cong F/S$ in which S is the preimage of N in F. The kernel of the natural map $\gamma_{c+1}^*(L) \longrightarrow \gamma_{c+1}^*(L/N)$ is then $\gamma_{c+1}(S, F)/\gamma_{c+1}(R, F)$. Therefore, it suffices to verify that $\gamma_{c+1}(S, F) = \gamma_{c+1}(R, F)$ if and only if $N \subseteq Z_c^*(L)$. Set $\bar{F} = F/\gamma_{c+1}(R, F)$, $\bar{R} = R/\gamma_{c+1}(R, F)$ and $\bar{S} = S/\gamma_{c+1}(R, F)$. Then $\gamma_{c+1}(S, F) = \gamma_{c+1}(R, F)$ is equivalent to $\bar{S} \subseteq Z_c(\bar{F})$. Invoking Corollary 2.2, $\bar{\pi}(Z_c(\bar{F})) = Z_c^*(L)$. Consequently, it may be inferred that $\bar{\pi}(\bar{S}) \subseteq Z_c^*(L)$ if and only if $\bar{S} \subseteq Z_c(\bar{F})$. Taking into account that $\bar{\pi}(\bar{S}) = N$, the result follows. $\square$

As an immediate consequence of Proposition 2.3 and [7, Corollary 2.2], we have

Corollary 2.4. *Let N be an ideal of a finite dimensional Lie algebra L which lies in $Z_c(L)$. Then $N \subseteq Z_c^*(L)$ if and only if* $\dim(\mathcal{M}^{(c)}(L/N)) = \dim(\mathcal{M}^{(c)}(L)) + \dim(N \cap \gamma_{c+1}(L))$.

Now, we are ready to prove Theorem A.

Proof of Theorem A. (i) By Lemma 1.2(ii) and Proposition 2.3, we obtain an epimorphism $\bigwedge^{c+1}(L/Z_c^*(L)) \longrightarrow \gamma_{c+1}^*(L/Z_c^*(L)) \cong \gamma_{c+1}^*(L)$. It is shown in [5] that the exterior product $M \wedge N$ of two crossed modules is finite dimensional if both M and N are finite dimensional. Consequently, if $L/Z_c^*(L)$ is finite dimensional then so is $\bigwedge^{c+1}(L/Z_c^*(L))$. The result follows.

(ii) It follows from [9, Theorem 2.2] and an argument similar to that used in the proof of part (i). $\square$

In readiness for the proof of Theorem B, we recall from [8] the concept of the universal central extension of a Lie algebra.

Let $e_i : 0 \longrightarrow M_i \longrightarrow K_i \xrightarrow{\theta_i} L \longrightarrow 0$, $i = 1, 2$, be central extensions of a Lie algebra L. Then we say that the extension e_1 covers (uniquely covers) the extension e_2 if there exists a homomorphism (or a unique homomorphism) $\phi_1 : K_1 \longrightarrow K_2$ such that $\theta_2\phi_1 = \theta_1$. Now, the central extension e_1 is called *universal* if it covers uniquely any central extension of L.

We have the following results regarding the universal central extension of perfect Lie algebras.

Lemma 2.5. *(See [8].) Using the above notations, the following statements hold*:

(i) *If the central extensions e_1 and e_2 are universal, then there is an isomorphism $K_1 \longrightarrow K_2$ that carries M_1 onto M_2.*
(ii) *If e_1 is universal, then K_1 and L are both perfect.*
(iii) *If K_1 is perfect, then e_1 covers e_2 if and only if e_1 uniquely covers e_2.*

Proposition 2.6. *(See [8].) Let L be a perfect Lie algebra. Then*

$$0 \longrightarrow \mathcal{M}(L) \longrightarrow \gamma_2^*(L) \xrightarrow{\mu_2} L \longrightarrow 0$$

is the universal central extension of L.

From the above conclusions and Proposition 1.3, we deduce that

Corollary 2.7. *For any perfect Lie algebra L, $0 \longrightarrow \mathcal{M}(L) \longrightarrow L \wedge L \xrightarrow{\kappa_L} L \longrightarrow 0$ is the universal central extension of L.*

Now we are able to prove Theorem B.

Proof of Theorem B. By virtue of Lemma 2.5(i), it is enough to show that the exact sequence $0 \longrightarrow \mathcal{M}^{(c)}(L) \longrightarrow \gamma_{c+1}^*(L) \xrightarrow{\mu_{c+1}} L \longrightarrow 0$ is the universal central extension of L for all $c \geqslant 1$. Owing to Proposition 2.6, the case $c = 1$ is true. Inductively, assume that the result holds for $c \geqslant 1$. Since $\gamma_{c+1}^*(L)$ is perfect, it follows from Corollary 2.7 that

$$e : 0 \longrightarrow \mathcal{M}\big(\gamma_{c+1}^*(L)\big) \longrightarrow \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \xrightarrow{\kappa_{c+1}} \gamma_{c+1}^*(L) \longrightarrow 0$$

is the universal central extension of $\gamma_{c+1}^*(L)$. Put $\delta = \mu_{c+1}\kappa_{c+1}$ and $B = \ker\delta$. We claim that the exact sequence $0 \longrightarrow B \longrightarrow \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \xrightarrow{\delta} L \longrightarrow 0$ is the universal central extension of L. For any $b \in B$, $\kappa_{c+1}(b) \in Z(\gamma_{c+1}^*(L))$ and then the image of the inner derivation map $ad_{\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)}(b)$ is central, whence the map is a homomorphism. As the central extension e is universal, Lemma 2.5(ii) indicates that the Lie algebra $\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)$ is perfect. Hence the map $ad_{\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)}(b)$ must be zero, implying $b \in Z(\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L))$. We therefore conclude that B is contained in the centre of $\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L)$. Now, assume that $0 \longrightarrow C \longrightarrow P \xrightarrow{\sigma} L \longrightarrow 0$ is an arbitrary central extension of L. Evidently, $T = \{(x, p) \in \gamma_{c+1}^*(L) \oplus P \mid \mu_{c+1}(x) = \sigma(p)\}$ is a subalgebra of $\gamma_{c+1}^*(L) \oplus P$ and $0 \longrightarrow 0 \oplus C \longrightarrow T \xrightarrow{\lambda} \gamma_{c+1}^*(L) \longrightarrow 0$ a central extension of $\gamma_{c+1}^*(L)$, in which λ denotes the natural projection. Thanks to the universality of the extension e, we can find a homomorphism $\alpha : \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \longrightarrow T$ with $\kappa_{c+1} = \lambda\alpha$. If $\gamma : T \longrightarrow P$ is the natural projection, then $\sigma(\gamma\alpha) = \delta$ and this proves our claim.

Therefore, by the induction hypothesis and Lemma 2.5(i), there exists an isomorphism $\varphi : \gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \longrightarrow \gamma_{c+1}^*(L)$ such that $\mu_{c+1}\varphi = \delta$. It is readily verified that the following diagram is commutative:

$$\begin{tikzcd}
\gamma_{c+1}^*(L) \wedge \gamma_{c+1}^*(L) \arrow[r, "1_{\gamma_{c+1}^*(L)} \wedge \mu_{c+1}"] \arrow[dr, "\varphi"'] & \gamma_{c+1}^*(L) \wedge L \arrow[r, "\kappa"] & \gamma_{c+2}^*(L) \arrow[dl, "\beta"] \\
& \gamma_{c+1}^*(L) &
\end{tikzcd}$$

where β is the canonical homomorphism and κ is homomorphism due to Lemma 1.2. As L is perfect, the crossed module μ is surjective. Consequently both $1_{\gamma^*_{c+1}(L)} \wedge \mu_{c+1}$ and κ are isomorphisms, implying that $\gamma^*_{c+2}(L)$ is isomorphic to $\gamma^*_{c+1}(L)$. This completes the induction and the proof of the theorem. $\square$

In [1], it is proved that the multiplier of a cover of a finite dimensional perfect Lie algebra is zero (also see [8]). Now, Theorem B shows this result for c-nilpotent multipliers of covers of finite dimensional perfect Lie algebras.

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